Extended-range ballistic aggregation: exact results and scaling

1985 J. Phys. A: Math. Gen. 18 611

(http://iopscience.iop.org/0305-4470/18/4/011)

J. Phys. A: Math. Gen. 18 (1985) 611-620. Printed in Great Britain

Extended-range ballistic aggregation: Exact results and scaling

P Gelband and P N Strenski

Department of Physics, Ginzton Laboratories, Stanford University, Stanford CA 94305, USA

Received 15 August 1984, in final form 3 October 1984

Abstract. We present the exact solution of an infinite-range ballistic aggregation model. The density is found to decrease as $N^{-1/2}$ as $N \to \infty$, where N is the number of columns in which particles move. Exact expresssions for the correlation function and the hole-size distribution are derived. Both decay exponentially over their entire domain. Using a scaling ansatz similar to that made in finite-size scaling in critical phenomena, we extrapolate these results to large but finite ranges. Computer simulations test the proposed scaling.

1. Introduction

Many models have been proposed recently to describe irreversible growth processes. Typically, growth is initialised with some number of seed particles; each model specifies how subsequent particles aggregate. In the Witten and Sanders (1981, 1983) model of diffusion-limited aggregation (DLA), a particle introduced from far off executes a random walk until it reaches a site adjacent to the aggregate and sticks, becoming part of the aggregate. A more general model, directed DLA, accounts for the possibility of an external field by allowing the random walk to occur with a different weight in each direction (Meakin 1983a, Jullien *et al* 1984, Nadal *et al* 1984). DLA is the limit that all weights are the same. The opposite extreme is the case in which particles move along straight lines in one direction. This model, known as ballistic aggregation or the 'rain' model (Bensimon *et al* 1983, 1984, Meakin 1983b), describes particles diffusing in a preferred direction with a very long mean-free path, for example, crystals grown *in vacuo* at low temperatures.

One common aspect of these models, and others like them, is that they are very simply stated but very difficult to solve. Most of what is known about these models has been learned through simulation, although there have been various mean-field (Muthukumar 1983, Tokuyama and Kawasaki 1984, Hentschel 1984, Ball *et al* 1984) and renormalisation-group (Gould *et al* 1983, Sahimi and Jerauld 1983) treatments proposed. In this paper we present a generalised 'rain' model which we call extended-range ballistic aggregation (ERBA). In this model particles 'stick' when they come within a specified lateral range of the aggregate. In § 2 we outline the exact solution of ERBA with infinite range. We calculate the density, correlation function, and hole-size distribution. In § 3 we make a scaling ansatz similar to that used in finite-size scaling (Fisher 1971, Fisher and Barber 1972, Barber 1982) to extrapolate these results to finite range. We present the results of simulations which test this ansatz.

0305-4470/85/040611+10$02.25

2. Exact results

2.1. The model

In the 'rain' model of ballistic aggregation (Bensimon *et al* 1983, 1984, Meakin 1983b), balls are dropped from random positions high above the substrate and fall straight downward. A given ball stops when it sticks to another ball aready in the aggregate. On a square lattice, a ball dropped in a given column will fall until it sees a particle either in a nearest-neighbour lattice site to the side of it, or in the lattice site directly below it. Then it sticks and becomes part of the aggregate.

We define the ERBA model as follows: let r be the range. Then in ERBA a falling particle sticks as soon as it sees an occupied site within r lattice spacings on the same level, or one in the lattice site immediately below it. The 'rain' model is the case $r = 1$.

In this paper we discuss N-column ERBA. We define the one- (two-) dimensional problem to be N semi-infinite columns arranged on a line (square). The problem in higher dimensions is defined analogously. The one-dimensional case for $r = 2$ is illustrated in figure 1. We note that in d-dimensional ERBA, a $(d+1)$-dimensional structure is grown. We assume periodic boundary conditions, and take for initial condition a seed particle at the bottom of each column. Subsequently, the height of a given column is defined as the distance above the seed particle (in units of the lattice spacing) of the most recently occupied site in the column.

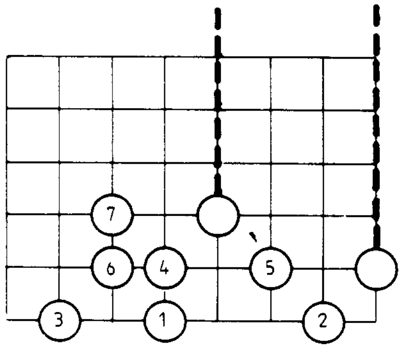

Figure 1. Possible growth sequence for ERBA with $r = 2$, $d = 1$. Numbers indicate the order in which balls were introduced into the lattice. Broken lines indicate possible trajectories for subsequent balls.

We call the case in which every column interacts with every other column the infinite-range model (IRM). In one dimension this occurs for $r \geq N/2$. Properties of the IRM are independent of dimension, because the spatial relationship of the columns to each other is irrelevant. We focus on this model first, since it is exactly solvable. The solution provides information, via scaling hypotheses, about the finite-range model.

The IRM grows row by row. This is because once the structure has grown to a certain height (defined as the height of the tallest column or columns), no balls dropped subsequently can stick below that height. The first site occupied in a given row must be directly above an occupied site in the previous row. The total number of balls in a completed row, however, is independent of that in any other row, and hence this structure has no boundary layer. This is in contrast to ERBA with finite range, including the 'rain' model, in which there is a period of transience during which the density relaxes to its bulk value.

2.2. *Solution for the number of tallest columns*

In this model, we drop one ball each time step. We define $P_t(n)$ to be the probability that at time t there are n tallest columns, i.e., there are n columns with height $h = \max_{j=1,\ldots,N}(h_j)$. Then the following recursion relations are an immediate consequence of the definition of the IRM:

$$
\begin{aligned}
P_{t+1}(1) &= \sum_{m=1}^{N} \frac{m}{N} P_t(m) \\
P_{t+1}(n) &= [(N-n+1)/N]P_t(n-1) \qquad (2 \leq n \leq N).
\end{aligned}
\tag{1}
$$

This system of equations describes a primitive, homogeneous Markov chain with a finite number of states (Gantmacher 1959). That is, the coefficients are independent of time, and for all i, j there exists a finite probability after N time steps of having j tallest columns given an initial state with i tallest columns. For such a system the asymptotic probabilities $P(n) \equiv \lim_{t\to\infty} P_t(n)$ are guaranteed to exist, and the approach to the limit is geometric. Specifically, for $t \to \infty$ we can remove the subscripts from (1). Together with the normalisation condition

$$\sum_{n=1}^{N} P(n) = 1 \tag{2}$$

these equations are easily solved. We find

$$P(n) = \frac{N^{N-n}}{(N-n)!} P(N) \qquad (1 \leq n \leq N-1) \tag{3a}$$

$$P(N) = \left(\sum_{m=0}^{N-1} \frac{N^m}{m!} \right)^{-1} \tag{3b}$$

where $P(N)$ is the probability of finding all columns at the same height. This description of the system in terms of the $P(n)$ will be useful in evaluating other quantities.

2.3. *Density*

Another quantity of interest is the probability of finding m balls in a given row once the row is complete, i.e., the next row has been started. This probability, which we call $Q(m)$, can be derived as follows: every completed row has at least one ball in it. After the first ball is in place in the row, the next ball dropped will fall on top of the first one with probability $1/N$, in which case the row is completed and a new row begun. Therefore $Q(1) = 1/N$. Otherwise, with probability $(1 - 1/N)$ there will be two balls in the row. The next ball dropped will fall on top of one of the first two with probability $2/N$, and thus $Q(2) = (1 - 1/N)2/N$. Continuing this process, we find the general expression

$$Q(m) = (1 - 1/N)(1 - 2/N) \ldots [1 - (m-1)/N]m/N, \tag{4}$$

which holds independently for every row of the structure. The average row density, or equivalently, the density of a structure that has been grown a long time, is

$$\rho = \frac{1}{N}\sum_{m=1}^{N} mQ(m). \tag{5}$$

There are two alternative ways to compute ρ which will prove useful. First, say a large number, M, of balls are dropped. Since every time the height of the structure increases by one there is then one tallest column, the height of the structure will be equal to the number of times there was one tallest column. This number is $MP(1)$. Therefore, the structure occupies a volume $NMP(1)$, and the density is

$$\rho = [NP(1)]^{-1}. \tag{6}$$

A third expression for ρ is obtained by noting that (3a) and (4) may be used to relate the distributions Q and P:

$$Q(m) = mP(m)/NP(1). \tag{7}$$

Then using equations (2), (6), and (7) we find

$$\rho = \sum_{m=1}^{N} \frac{Q(m)}{m}. \tag{8}$$

2.4. Asymptotic behaviour of the density

We now derive the asymptotic behaviour $\rho \sim \mathrm{O}(N^{-1/2})$ as $N \to \infty$. First we show that $P(N) \sim \mathrm{O}(\mathrm{e}^{-N})$ as $N \to \infty$. $P(N)^{-1}$ is simply the first N terms in a Taylor expansion of e^N (see equation (3b)). The second N terms of this expansion, taken in reverse order, are equal to the first N except for corrections which are unimportant as $N \to \infty$. We have

$$\begin{aligned}
\mathrm{e}^N &= \left(\sum_{m=0}^{N-1} + \sum_{m=N}^{2N-1} + \sum_{m=2N}^{\infty}\right)\frac{N^m}{m!} \\
&= P(N)^{-1} + \sum_{m'=0}^{N-1} \frac{N^{m'}}{m'!} \prod_{n=0}^{N-m'-1} \frac{N^2}{N^2 - n^2} + \sum_{m=2N}^{\infty} \frac{N^m}{m!} \\
&= 2P(N)^{-1} + \sum_{m=1}^{N} \frac{N^{N-m}}{(N-m)!}\left[\prod_{n=0}^{m-1}\left(\frac{N^2}{N^2-n^2}\right) - 1\right] + \sum_{m=2N}^{\infty}\frac{N^m}{m!}.
\end{aligned} \tag{9}$$

The last term in (9) is bounded as follows:

$$\begin{aligned}
\sum_{m=2N}^{\infty} \frac{N^m}{m!} &= \frac{N^{2N}}{(2N)!}\left(1 + \frac{N}{2N+1} + \frac{N^2}{(2N+2)(2N+1)} + \ldots\right) \\
&< 2N^{2N}/(2N)! \\
&\sim \mathrm{e}^{2N}2^{-2N}(\pi N)^{-1/2} \qquad \text{as } N \to \infty \\
&< \mathrm{e}^N N^{-1/2}
\end{aligned}$$

where we have approximated $(2N)!$ using Stirling's formula. Therefore, this term does not contribute to the leading asymptotic behaviour of $P(N)$ as $N \to \infty$.

We now consider the second term in (9). It can be shown that the largest of the N summands in this term occurs for $m \sim (3N)^{1/2}$ as $N \to \infty$. With the aid of the identity $\lim_{n\to\infty}(1+x/n)^n = \mathrm{e}^x$ and Stirling's formula, we find that this largest summand is asymptotically $O(e^N N^{-3/2})$. Thus the second term in (9) is at most $O(e^N N^{-1/2})$, and so $P(N) \sim 2e^{-N}$. Using (3*a*) we have $P(1) \sim (2/\pi)^{1/2} N^{-1/2}$. The density, using (6), is then

$$\rho \sim (\pi/2)^{1/2} N^{-1/2} \qquad (N \to \infty). \tag{10}$$

The velocity of growth, defined as the average change in the height of the structure every N time steps, is simply the inverse of the density, and so diverges as $N^{1/2}$ as $N \to \infty$.

2.5. *Correlation function*

As discussed in § 2.1, structures in the IRM grow row by row. A new row is begun by the first particle that falls on a site occupied in the row just below it. At subsequent time steps particles fill in the new row at random. Thus, only the placement of the first particle in each row contains information about the previous row, because the first particle is positioned directly above a particle in that row. This is the sole source of correlations in the IRM.

The density-density correlation function is defined as

$$c_{ij}(h) \equiv \langle n_i(0) n_j(h) \rangle - \rho^2$$

where $n_k(h)$ is the occupation (0 or 1) of the kth site in row h, and $\langle \ldots \rangle$ represents an average over all possible structures grown according to the rules. Since these structures have no boundary layer, 0 refers to any reference row provided row h is complete.

First we consider $c_{ii}(h)$, the correlation function for sites in the same column. $c_{i \neq j}(h)$ will follow trivially. Let $l(l')$ be the first site occupied in row 1 (row h) and let $Z(h-1)$ be the probability that $l = l'$. A recursion relation for $Z(h)$ may be constructed by considering the case that the completed row $h-1$ is occupied by m particles. If the first site occupied in row $h-1$ is site l, then $l' = l$ with probability $1/m$. Otherwise, site l in row $h-1$ is filled in subsequently with probability $(m-1)/(N-1)$, and then with probability $1/m$, row h will begin at this site. Therefore,

$$Z(h) = Z(h-1) \sum_{m=1}^{N} \frac{Q(m)}{m} + [1 - Z(h-1)] \sum_{m=1}^{N} \frac{Q(m)}{m} \frac{m-1}{N-1}. \tag{11}$$

Using the boundary condition $Z(0) = 1$ and equation (8) to evaluate the sums in (11) we find

$$Z(h-1) = \left(\frac{N-1}{N}\right)\left(\frac{\rho N - 1}{N-1}\right)^{h-1} + \frac{1}{N}. \tag{12}$$

If $l = l'$, then both row 0 and row h have site l occupied. If k and k' are the number of balls in row 0 and row h respectively upon completion of these rows, then the expected total number of sites, $\mathcal{N}_s(k, k')$, occupied both in row 0 and row h is $1 + (k-1)(k'-1)/(N-1)$. Defining $f_{ll'}$ to be the correlation function for given l, l' we have

$$f_{ll} = \frac{1}{N} \sum_{k,k'} Q(k) Q(k') \mathcal{N}_s(k, k') - \rho^2 = \frac{(1-\rho)^2}{N-1}. \tag{13}$$

To derive $f_{l \neq l'}$, we consider a related model in which the distribution of numbers k in completed rows is again $Q(k)$, but now the k particles in a given row are randomly placed. Note that in this model the first particle in a row need not be placed above a particle in the previous row. In this random model $\langle n_i(0)n_i(h)\rangle_{\mathrm{ran}} = \langle n_i(0)\rangle\langle n_i(h)\rangle = \rho^2$. This random occupation can occur in two ways: the first ball placed in row h occupies a given site that is also occupied in row 0, or it does not. In the first case, $\langle n_i(0)n_i(h)\rangle_{\mathrm{ran}} = f_{ll} + \rho^2$. In the second case, $\langle n_i(0)n_i(h)\rangle_{\mathrm{ran}} = f_{l\neq l'} + \rho^2$. Since for fixed l there are $N-1$ ways to choose $l' \neq l$, then

$$\langle n_i(0)n_i(h)\rangle_{\mathrm{ran}} = \frac{1}{N}(f_{ll}+\rho^2) + \frac{N-1}{N}(f_{l\neq l'}+\rho^2).$$

Thus we can solve for $f_{l\neq l'}$:

$$f_{l\neq l'} = -[1/(N-1)]f_{ll}. \tag{14}$$

The functions $f_{ll'}$ are identical for the two models: in each model, after the first ball is in place in a given row, subsequent balls are placed at random. Therefore, (14) also holds in the IRM.

Finally, using equations (12)-(14), we find that the correlation functions for sites within a single column is an exact exponential:

$$\begin{aligned} c_{ii}(h) &= Z(h-1)f_{ll} + [1-Z(h-1)]f_{l\neq l'} \qquad (h\geqslant 1) \\ &= \frac{(1-\rho)^2}{\rho N-1}\exp(-h/\xi) \qquad \text{where } \xi = \left[\ln\left(\frac{N-1}{\rho N-1}\right)\right]^{-1}. \end{aligned} \tag{15}$$

In the limit of large N, the correlation length vanishes: $\xi \sim 2/\ln N$. Correlations decay rapidly, as expected.

Using reasoning similar to that which led to (14), we find

$$c_{ii}(h) + (N-1)c_{i\neq j}(h) = 0.$$

Therefore, using (15), the correlation function for sites in different columns is

$$c_{i\neq j}(h) = \frac{-(1-\rho)^2}{(N-1)(\rho N-1)}\exp(-h/\xi) \qquad (h\geqslant 1).$$

This anticorrelation makes sense: a particle occupying column i in row 0 makes it more likely that there will be a particle in column i at row h, because row 1 (and then row 2, etc) may begin in that column. However, the average density ρ is the same for every row. So if it is more likely to find a particle in column i at row h, it is less likely to find a particle in column $j \neq i$ at row h.

2.6. *Hole-size distribution*

We now derive the hole-size distribution $\Omega(m)$ of a structure with an infinite number of completed rows. Since the placement of particles with respect to each other in a given row is random, we consider holes that exist within single columns only. Thus, we define the probability $\Omega(m)$ of finding a hole of size m as follows. Given a particular occupied site (defined to lie in row 0), $\Omega(m)$ is the probability that the first m sites above this site are vacant, while the $(m+1)$th site is occupied. The probability of a hole size 1 or greater is then

$$\Omega_N(m) = \left\langle \left(\frac{k_0}{\rho N}\right)\left(\frac{k_0-1}{k_0}\right)\left(\frac{N-k_1}{N-1}\right)\cdots\left(\frac{N-k_m}{N-1}\right)\left(\frac{k_{m+1}-1}{N-1}\right)\right\rangle \qquad (m\geqslant 1) \tag{16}$$

where k_r is the number of occupied sites in row r upon completion of the row, and $\langle\ldots\rangle$ signifies an average over the $Q(k_r)$.

The first factor in equation (16), $k_0/\rho N$, is the fraction of occupied sites that are in a row with k_0 occupied sites total. The next factor, $(k_0-1)/k_0$, is the probability that a given occupied site in row 0, which we denote as site i, is not the first site occupied in row 1. The next m factors give the probability that no balls fall into column i while rows 1 through m are built up. The final factor, $(k_{m+1}-1)/(N-1)$, is the probability that a ball fills in column i at row $m+1$. For $m=0$ there is an additional contribution from the case that site i is the first site occupied in row 1. This happens with probability $1/\rho N$.

The k_r are independent quantities. Using (5) to evaluate the averages in (16) we find

$$\Omega_N(m)=\frac{(\rho N-1)^2}{\rho N(N-1)}\left(\frac{N-\rho N}{N-1}\right)^m+\frac{1}{\rho N}\,\delta_{m,0}. \tag{17}$$

The average hole size computed with this distribution is $\rho^{-1}-1$ (as it must be), and so increases as $N^{1/2}$ as $N\to\infty$. The hole-size distribution decays exponentially with a characteristic length also of order $N^{1/2}$, much larger than the correlation length ξ (equation (15)). This is not contradictory. For example, a totally random placement of balls on the lattice has a characteristic hole size while the correlation length is zero.

In fact, the probability of a hole of size m for this totally random distribution of balls with density ρ is:

$$\Omega_{\text{ran}}(m)=\rho(1-\rho)^m.$$

Comparing this distribution to that of equation (17) we find

$$\Omega_N(m)/\Omega_{\text{ran}}(m)=1+(1/\rho^2N)\,\delta_{m,0}+O(N^{-1/2}).$$

Aside from holes of size 0 (through which, as noted previously, the only information about the structure is passed) the distribution of hole sizes in the IRM is nearly random.

3. Scaling to finite range

ERBA with finite range has not been solved. However, by postulating a scaling ansatz similar to that made in finite-size scaling (FSS) in critical phenomena, we extrapolate non-trivial information about the finite-range problem from the exact solution of the IRM. Similar methods have been applied to other problems in aggregation (Rácz and Vicsek 1983, Turban and Debierre 1984).

3.1. Finite-size scaling in critical phenomena

The primary postulate of FSS (Fisher 1971, Fisher and Barber 1972, Barber 1982) is that near the bulk critical temperature T_c, the behaviour of a large but finite system is determined by the variable $L/\xi(T)$, where L is the characteristic linear dimension of the system and $\xi(T)$ is the bulk correlation length. In particular, if $\mathcal{Q}$ is a thermodynamic quantity exhibiting an algebraic divergence in the bulk as T_c is approached, then it is proposed that in the finite system $\mathcal{Q}$ has the form

$$\mathcal{Q}\sim L^{\mu}g(L/\xi(T)) \qquad (L,\,\xi\text{ large}). \tag{18}$$

If Q diverges with an exponent α and ξ with an exponent ν, then it is shown that this hypothesis is consistent with

$$\mathcal{Q} \sim L^{\alpha/\nu} \qquad (L \text{ large}, \xi = \infty, T = T_c). \tag{19}$$

The predictions of FSS have been confirmed in many examples in critical phenomena (Sur *et al* 1976, Barber 1982, Brezin 1982).

3.2. *Finite-range scaling*

In the same vein, we postulate that for ERBA with range r, the density has the form

$$\rho \sim r^{\mu} g(r/L) \qquad (r, L \text{ large}) \tag{20}$$

where $L \equiv N^{1/d}$ and d is the dimension. Note that L here plays the role of ξ in equation (18). When $r = L$ we have the IRM result $\rho \propto N^{-1/2}$ (see equation (10)). Therefore,

$$\mu = -d/2. \tag{21}$$

To arrive at a scaling form analogous to (19) we need to consider $g(0)$, which is obtained by evaluating

$$\lim_{r/L \to 0} \lim_{\substack{r, L \to \infty \\ r/L \text{ fixed}}} \rho r^{-\mu}.$$

If it is valid to change the order in which limits are taken, i.e., if $g(0)$ may be correctly evaluated by letting $r/L \to 0$ before letting $r \to \infty$, then we expect $g(0)$ to be finite. This is because, on the bais of our computer simulations, we expect that the density does not vanish for fixed r letting $L \to \infty$. If $g(0)$ is finite then

$$\rho \sim r^{-d/2} \qquad (r \text{ large}, L = \infty). \tag{22}$$

In FSS it is the same sort of switching of limits by which (19) is derived from (18). If it is not valid to switch the limits, then $g(0)$ may be 0. In this case we could expect

$$\rho r^{d/2} \sim (r/L)^{\sigma} \qquad (1 \ll r \ll L) \tag{23}$$

where σ is some positive power.

A similar argument for the correlation function, based on (15), would suggest

$$c_{ii}(h) \sim r^{-d/2} g_1(r/L, h \ln L) \qquad (r \text{ large}, L^{d/2} \gg h)$$

where h has been scaled by the correlation length of the IRM. For the hole-size distribution, using equation (17), we expect

$$\Omega_N(m) \sim r^{-d/2} g_2(r/L, m/L^{d/2}) \qquad (r \text{ large}, L^d \gg m)$$

where m has been scaled by the characteristic hole size in the IRM.

3.3. *Results of computer simulations*

We have tested the scaling hypothesis for the density (equations (20) and (21)) with $d = 1$ by running Monte Carlo simulations in which ERBA structures were grown to maximum heights ranging between 6000 and 50 000. We note that the scaling hypothesis is confirmed if, for fixed values of r/L, the function $\rho r^{1/2}$ goes to a finite constant $(= g(r/L))$ as $r \to \infty$. In figure 2 we plot our data for $r/L = \frac{1}{4}$. It appears in this figure

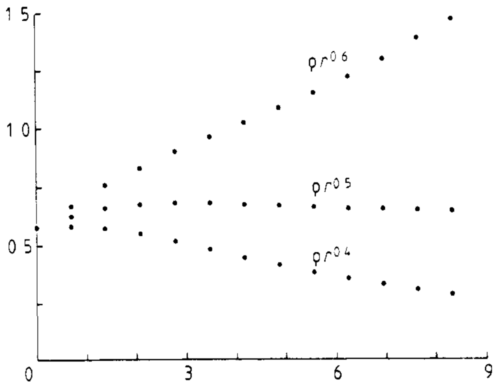

Figure 2. Test of leading power-law behaviour for scaling of the density ($\mu = -\frac{1}{2}$) as explained in the text.

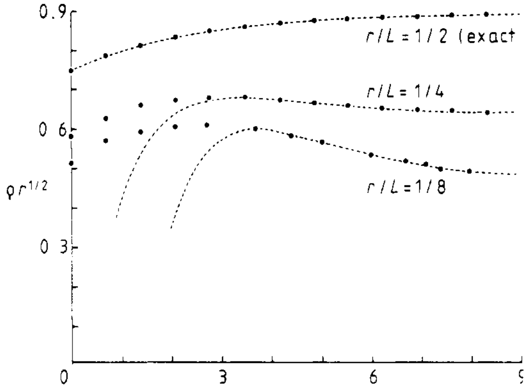

Figure 3. Results of computer simulations for $r/L = \frac{1}{4}$ and $\frac{1}{8}$; exact results for $r/L = \frac{1}{2}$. Least-squares fits to tails (see text) are shown.

that $\rho r^{0.4}$ vanishes and $\rho r^{0.6}$ diverges in the limit of large r. If this is correct, then for some intermediate exponent there is scaling (at least for $r = L = \frac{1}{4}$). The exponent $\frac{1}{2}$ is a likely candidate, in agreement with (20) and (21).

In figure 3 we plot our numerical values of $\rho r^{1/2}$ for several values of r/L (for $r/L = \frac{1}{2}$ the results are exact). We also show the results of a least-squares fit of the last eight data points on each curve to the form $g(r/L) + a_1(r/L)r^{-1/2} + a_2(r/L)r^{-1}$. These fits predict $g(\frac{1}{4}) = 0.627$ and $g(\frac{1}{8}) = 0.456$, both well above zero. For comparison, the fit for $r/L = \frac{1}{2}$ predicts the exact value $g(\frac{1}{2}) = \pi^{1/2}/2$ to five places. We have chosen the above functional form for two reasons. First, analysis strongly suggests that $\rho r^{1/2}$ for $r/L = \frac{1}{2}$ is a power series in $r^{-1/2}$. Second, FSS predicts that corrections to scaling are power law (Barber 1982).

We have also done simulations for smaller values of r/L. The data indicate that the scaling function g is an increasing function. However, because of slow convergence it has not been feasible to obtain g accurately, especially for small r/L. (For example, some of our data points required over 20 hours of CPU time on a VAX 11/780.) In particular, we have not been able to extrapolate to the limit $r/L \to 0$ to determine whether $g(0)$ is finite. Thus we were unable to identify the correct form of the density (either (22) or (23)) in the infinite-system limit.

We have not tested scaling for $d > 1$, or done rigorous testing of the correlation function or hole-size distribution. However, we have seen that $\Omega(0)$ scales are expected for $d = 1$. Thus, the scaling hypotheses of § 3.2 are completely consistent with the results of our simulations.

4. Discussion

For r large the underlying lattice structure should be unimportant. As a result, in this limit, the scaling functions of § 3.2 depend only on ratios of the relevant length scales. This reduction in the number of variables together with the power-law scaling behaviour observed in ERBA seems indicative of an underlying renormalisation group with fixed point at $r = L = \infty$. The possibility $g(0) = 0$ would correspond to the presence of a so-called 'dangerous irrelevant variable'. At this time, however, we do not understand the mechanics of this renormalisation group.

We also note that the IRM is mean-field like. This is because every column 'sees' every other column, and because all quantities computed are averages over many time steps. However, the IRM is not a mean-field theory of the 'rain' model. Because we do not decrease the strength of the interaction with increasing range, the density as $N \to \infty$ of the IRM vanishes, while that of the 'rain' model does not.

Acknowledgments

We wish to thank R M Bradley, E Domany, S Doniach, and M Gabay for helpful discussions. This research was supported by ONR grant N000-14-82-K-0524 and by NSF grant DMR-83-05723.

References

Ball R, Nauenberg M and Witten T A 1984 *Phys. Rev.* A **29** 2017
Barber M N 1982 *University of New South Wales preprint*
Bensimon D, Domany E and Aharony A 1983 *Phys. Rev. Lett.* **51** 1394
Bensimon D, Shraiman B and Liang S 1984 *University of Chicago preprint*
Brezin E 1982 *J. Physique* **43** 15
Fisher M E 1971 in *Critical Phenomena, Proc. Int. School of Physics 'Enrico Fermi', Varenna 1970, Course No 51* ed M S Green (New York: Academic)
Fisher M E and and Barber M N 1972 *Phys. Rev. Lett.* **28** 1516
Gantmacher F R 1959 *Applications of the Theory of Matrices* (New York: Interscience) pp 96–118
Gould H, Family F and Stanley H E 1983 *Phys. Rev. Lett.* **50** 686
Hentschel H G E 1984 *Phys. Rev. Lett.* **52** 212
Jullien R, Kolb M and Botet R 1984. *J. Physique* **45** 395
Meakin P 1983a *Phys. Rev.* B **28** 5221
—— 1983b *J. Colloid Interface Sci.* **96** 415
Muthukumar M 1983 *Phys. Rev. Lett.* **50** 839
Nadal J P, Derrida B and Vannimenus J 1984 *Phys. Rev.* B **30** 376
Rácz Z and Vicsek T 1983 *Phys. Rev. Lett.* **51** 2382
Sahimi M and Jerauld G R 1983 *J. Phys. A: Math. Gen.* **16** L419
Sur A, Lebowitz J L, Marro J, Kalos M H and Kirkpatrick S 1976 *J. Stat. Phys.* **15** 345
Tokuyama M and Kawasaki K 1984 *Phys. Lett.* **100A** 337
Turban L and Debierre J-M 1984 *J. Phys. A: Math. Gen.* **17** L289
Witten T A and Sander L M 1981 *Phys. Rev. Lett.* **47** 1400
—— 1983 *Phys. Rev.* B **27** 5686